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The Exact Condition of the B-Spline Basis May Be Hard to Determine

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The author's conjecture concerning the knot sequence whose associated B-spline sequence has maximum max-norm condition number is disproved. Related condition numbers are explored and the corresponding conjecture concerning the "worst" knot sequence for them is further supported by numerical results. © 1990 Academic Press, Inc.

At the end of a long discussion of the linear functionals which vanish at all B-splines but one in [B2], I conjectured that $D_{k,\infty}$, the worst possible condition with respect to the max-norm of a B-spline basis of order k, occurs when the knots have high multiplicity. I went further than that on p. 155 of [B3], where I displayed supposed values of $D_{k,\infty}$ based on this conjecture. The conjecture was based in good part on detailed calculations of a closely related problem in [B1], on a calculation of the number D_k which provides a bound for the worst B-spline condition with respect to any p-norm, and on some calculations of the max-norm condition itself. In particular, I wrote: "As with the earlier reported calculations of D_k, it appears from these calculations that" the worst condition "is taken on at the 'middle' vertex of the simplex" of knot sequences over which the maximization takes place. "This would mean that

$$D_{k,\infty} = \|(N_{j,k,\tau}(\rho_i))^{-1}\|_\infty$$

with $\tau := (\tau_i)_1^{2k}$ given by

$$0 = \tau_1 = \cdots = \tau_k, \qquad \tau_{k+1} = \cdots = \tau_{2k} = 1$$

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and $0=\rho_1<\cdots<\rho_k=1$ the extrema of the Chebyshev polynomial of degree $k-1$ for $[0, 1]$. This gives the following values for $D_{k,\infty}$: $\cdots$." In other words, I conjectured that the worst max-norm condition occurs for a knot sequence without interior knots and, assuming this to be true, computed and displayed this condition number for the first few values of the order k as the value of $D_{k,\infty}$.

Note that $\|(N_{j,k,\tau}(\rho_i))^{-1}\|_\infty=\|a\|_\infty$ with $C_\tau:=\sum_j N_{j,k,\tau}a(j)$ the unique spline satisfying $C_\tau(\rho_j)=(-)^{k-j}$, $j=1, ..., k$. This implies that C_τ is the Chebyshev polynomial of degree $k-1$ for the interval $[0, 1]$ and the numbers computed and displayed as $D_{k,\infty}$ in [B2, B3] are therefore the absolutely largest coefficients in the expansion of the Chebyshev polynomial as a linear combination of the B-splines $N_{j,k,\tau}$. Because of the special nature of the knot sequence τ, these B-splines reduce, on the interval $[0, 1]$ of interest, to the polynomials in the Bernstein form. This led Lyche [L] to the observation that there was no need for numerical calculations since the Bernstein form for the Chebyshev polynomial could be written out explicitly and a simple expression for its absolutely largest coefficient could be provided. Because of this connection, I shall refer to the knot sequence without interior knots more briefly as the *Bernstein knots*. The explicit formula allowed Lyche [L] to verify my conjecture that this condition number grows like 2^k.

Since then, there have been several attempts at verifying the conjecture that the worst max-norm condition is had by the Bernstein knots. It is therefore important to point out with the aid of specific examples that the conjecture is incorrect in general. In contrast, more detailed calculations concerning the related number

$$D_k := \sup_{\mathbf{t}} \sup_j \sup_a |a(j)|\ |I_j| \Big/ \int_{I_j} \Big|\sum_i N_{i,k,\mathbf{t}} a(i)\Big|$$

(with $I_j := [t_j, t_{j+k}]$ the support of N_j) have so far failed to shake the corresponding conjecture that D_k is attained by the Bernstein knots.

Condition Number Defined

It is convenient to define the *condition number* cond of the basis (φ_i) of a normed linear space S as the number

$$\text{cond} := \sup_a \frac{\|\sum_i \varphi_i a(i)\|}{\|a\|_\infty} \sup_a \frac{\|a\|_\infty}{\|\sum_i \varphi_i a(i)\|}.$$

For, assuming the basis (φ_i) so normalized that the first supremum is 1, this gives the equality

$$\text{cond} = \sup_j \|\lambda_j\|,$$

with

$$\lambda_j : \sum_i \varphi_i a(i) \mapsto a(j)$$

the jth coordinate functional for the basis (φ_i).

Let $\mathbf{t} := (t_i)$ be a knot sequence for splines of order k, i.e., $t_i < t_{i+k}$, all i, and let (N_i) be the corresponding (normalized) B-spline basis for the spline space $S := S_{k,\mathbf{t}}$ (see, e.g., [B3], for relevant definitions and details). The N_i are nonnegative and sum to 1, hence

$$\sup_a \frac{\|\sum_i N_i a(i)\|_\infty}{\|a\|_\infty} = 1,$$

where here and below we take

$$\|f\|_\infty := \sup_{t_k \leqslant t \leqslant t_{n+1}} |f(t)|$$

in case $\mathbf{t}$ is finite. Denote by $\lambda_i = \lambda_{i,\mathbf{t}}$ the ith coordinate functional for this basis and by

$$\text{cond}_{k,\mathbf{t}} := \sup_a \frac{\|a\|_\infty}{\|\sum_i N_i a(i)\|_\infty} = \sup_i \|\lambda_{i,\mathbf{t}}\|$$

its *condition number.*

A Counterexample

Let f be the even piecewise cubic on $[-1, 1]$ with just one breakpoint, at 0, given by

$$f(x) := \begin{cases} T_3(1 + (1-\alpha)(x-1)), & x \geqslant 0; \\ f(-x), & x \leqslant 0, \end{cases}$$

with $T_3 = 4(\ )^3 - 3(\ )$ the cubic Chebyshev polynomial and $\alpha := -\frac{1}{2}$ its negative extreme point (see Fig. 1). Since $Df(0+) = 0$, f is in C^2, i.e., a cubic spline with a simple knot, at 0. One readily computes its cubic B-spline coefficients (for the knot sequence $\mathbf{t} := (-1, -1, -1, -1,$

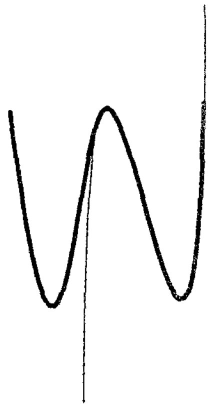

FIG. 1. A cubic Chebyshev spline with one knot constructed from the cubic Chebyshev polynomial.

0, 1, 1, 1, 1)) to be $(1, -\frac{7}{2}, \frac{11}{2}, -\frac{7}{2}, 1)$. Since $\|f\|_\infty = 1$, this implies that $\text{cond}_{4,\mathbf{t}} \geqslant 5.5$, while the cubic Bernstein-knots condition number is 5 (see [B2, L]).

A LEMMA

The number

$$D_{k,\infty} := \sup_{\mathbf{t}} \sup_{i} 1/\text{dist}_{\infty, [t_{i+1}, t_{i+k-1}]}(N_i, \text{span}(N_j)_{j \neq i})$$

was introduced in [B2] as a convenient upper bound for the worst B-spline condition number

$$\text{cond}_k := \sup_{\mathbf{t}} \text{cond}_{k,\mathbf{t}}.$$

The following lemma shows that the two numbers are equal, hence that $\text{cond}_k = D_{k,\infty}$ can be determined by *local* means.

LEMMA 1.

$$\sup_{\mathbf{t}} \sup_{i} \|\lambda_i\| = \sup_{\mathbf{s}} \|\lambda_{k-1,\mathbf{s}}\|_I,$$

where $\mathbf{s}$ *is any knot sequence of the particular type*

$$s_1 = \cdots = s_k = -1 \leqslant s_{k+1} \leqslant \cdots \leqslant s_{2k-3} \leqslant 1 = s_{2k-2} = \cdots = s_{3k-3}$$

and $\|\lambda\|_I := \sup_f |\lambda f| / \|f\|_I$ *with* $\|f\|_I$ *the max-norm on* $I := [-1, 1]$.

Proof. It is sufficient to prove that, for any $\mathbf{t}$ and any i,

$$\|\lambda_i\| \leqslant D := \sup_{\mathbf{s}} \|\lambda_{k-1,\mathbf{s}}\|_I.$$

Since $\lambda_i(\ )^0 = 1$ for any i, we have $\min_i \|\lambda_i\| \geqslant 1$ and, in particular, $D \geqslant 1$. If $t_{i+1} = t_{i+k-1}$, and without loss of generality, $t_i < t_{i+1}$, then $\lambda_i f = f(t_{i+1})$, hence $\|\lambda_i\| = 1 \leqslant D$, and we are done in this case.

In the contrary case, $t_{i+1} < t_{i+k-1}$, hence, after a linear change of the independent variable, we may assume that $t_{i+1} = -1$, $t_{i+k-1} = 1$. Now let $\mathbf{t}'$ be the knot sequence obtained from $\mathbf{t}$ by inserting both -1 and 1 enough times to increase their multiplicity to $k-1$ and let i' be such that $t'_{i'+j} = t_{i+j}$ for $j = 1, \ldots, k-1$. Then, with λ'_j the jth coordinate functional for the basis $(N_{i,k,\mathbf{t}'})$ of the refined spline space of the same order,

$$\|\lambda_i\| \leqslant \|\lambda'_{i'}\|,$$

since the (now standard) formula (cf., e.g., p. 116 of [B3])

$$\lambda_i f = \sum_{r<k} (-D)^{k-1-r} \psi(\xi_i)\, D^r f(\xi_i),$$

with $\xi_i \in\ ]t_i, t_{i+k}[$ and

$$\psi := (t_{i+1} - \cdot) \cdots (t_{i+k-1} - \cdot)/(k-1)!,$$

shows that $\lambda_i f$ only involves the knots t_{i+j}, $j = 1, \ldots, k-1$, hence $\lambda_i f = \lambda'_{i'} f$ for all $f \in S$. This finishes the proof since

$$\|\lambda'_{i'}\| = \sup_f |\lambda'_{i'} f| / \|f\|_\infty \leqslant \sup_f |\lambda'_{i'} f| / \|f\|_I \leqslant D. \quad \blacksquare$$

COMPUTATION OF $D_{k,\infty}$

According to Lemma 1, $D_{k,\infty}$ is the maximum of the function

$$d: [-1, 1]^{k-3} \to \mathbb{R}: \sigma \mapsto \|\lambda_{k-1,\mathbf{s}}\|_I$$

with $(s_{k+i})_{i=1}^{k-3}$ the sequence of "interior" knots of the knot sequence $\mathbf{s} = (s_i)_1^{3k-3}$ in $[-1, 1]$ obtained from σ by ordering. The failed conjecture amounts to the statement that the maximum is taken on at the "middle" vertex of the domain of d.

For $k = 3$, there are no interior knots and, correspondingly, $D_{3,\infty} = 3$, the condition number of the Bernstein-knot B-splines.

For $k = 4$, there is just one interior knot, hence the calculation of $D_{k,\infty}$ amounts to the maximization of the function $d(\xi)$ as ξ traverses the interval

$[-1, 1]$. A drawing of this function is available in Fig. 2; it is the hindmost curve. This shows that d has a local *minimum* at $\xi = 0$ (necessarily a critical point because of symmetry), and that the maximum (and, at least numerically, a good estimate for $D_{4,\infty}$) is 5.5680..., which occurs when $\xi \sim \pm 0.472$.

It is, in some sense, not too surprising that, for $k = 4$, the maximum occurs in the interior rather than at a vertex, since, after all, $\xi = 0$ is necessarily a critical point, by symmetry, and there are, correspondingly, two "middle" vertices. It is much more discouraging that, for $k = 5$, the maximum is also taken on at an interior point, for, in this case, there is only one "middle" vertex. Figure 3 shows d as a function of the two interior knots. According to [B2, L], the max-norm condition in the quartic case for the Bernstein knots is 11.666... But one computes in this case that $D_{5,\infty} \sim 12.088$ and this occurs when the two interior knots, both simple, are at the symmetric points $\sim \pm 0.89$.

The sharp drop toward the boundary values is an indication of the general situation. Numerical experimentation for $k \leqslant 8$ seems to indicate that, for $k > 3$, the maximum occurs at an **s** close to, but not at, a vertex, with d rising sharply initially as one moves away from the boundary. For odd k, the maximum seems to occur near the "middle" vertex. For even k, it occurs at a point (two points for small k) near what passes for the "middle" vertex in that case, i.e., with both 0 and 1 knots of the same multiplicity and $\frac{1}{2}$ a simple knot.

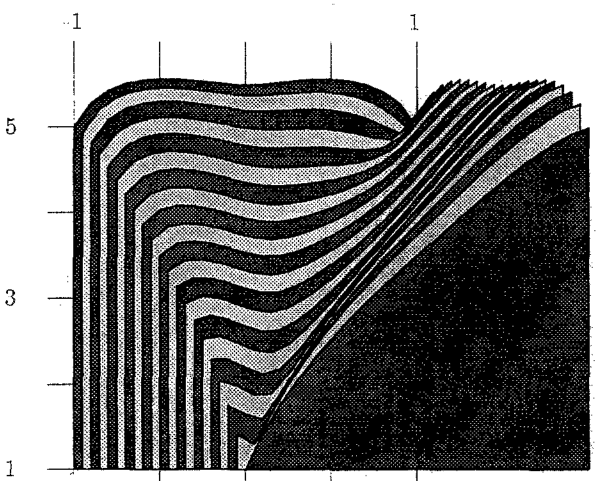

FIG. 2. Condition number of cubic B-splines with two interior knots. Sections are shown corresponding to one knot fixed while the other traverses $[-1, 1]$. As the "fixed" knot traverses $[-1, 1]$, the corresponding section is increasingly offset to provide "insight" into the surface.

For the evaluation of $\|\lambda_j\|_\infty$, consider the "Chebyshev spline" $C_{\mathbf{s}}$ for the knot sequence $\mathbf{s}$, i.e., $C_{\mathbf{s}} \in S := S_{k,\mathbf{s}}$, of max-norm 1, and *maximally alternating*, i.e., there is an increasing sequence $(\rho_j)_1^n$ (with $n := \dim S$) so that $C_{\mathbf{s}}(\rho_j) = (-)^{n-j}$, all j. (It can be shown that such $C_{\mathbf{s}}$ exists, and uniquely so; see, e.g., [M]). Let c be the sequence of its B-spline coefficients. This sequence necessarily strictly alternates in sign at least $n-1$ times, hence all $c(j)$ are nonzero. This implies that, for each j, $C_j := C_{\mathbf{s}}/c(j)$ is well defined and in $N_j + \operatorname{span}(N_i)_{i \neq j}$, therefore necessarily the error in the best uniform approximation to N_j from $\operatorname{span}(N_i)_{i \neq j} = \ker \lambda_{j,\mathbf{s}}$, hence an extremal for $\lambda_{j,\mathbf{s}}$, and therefore $\|\lambda_{j,\mathbf{s}}\|_\infty = 1/\|C_j\|_\infty = |c(j)|$. This reduces the evaluation of the function d to be maximized to the numerical construction of the Chebyshev spline, as is done in the following MATLAB (cf. [MBLK]) script.

```
function [sp, rho, a, iter] = chebmk(t, k, rho)
%
%        [sp, rho, a, iter] = chebmk(t, k[, rho])
%
% returns the Chebyshev spline for the given knot sequence t_1, ..., t_n + k,
% as well as the sequence rho of its alternating points and the sequence
% a of its B-spline coefficients. On input, rho is assumed to contain a
% reasonable first guess. If missing, the knot averages are used.
%
%   By definition, the Chebyshev spline is the unique linear combination
```

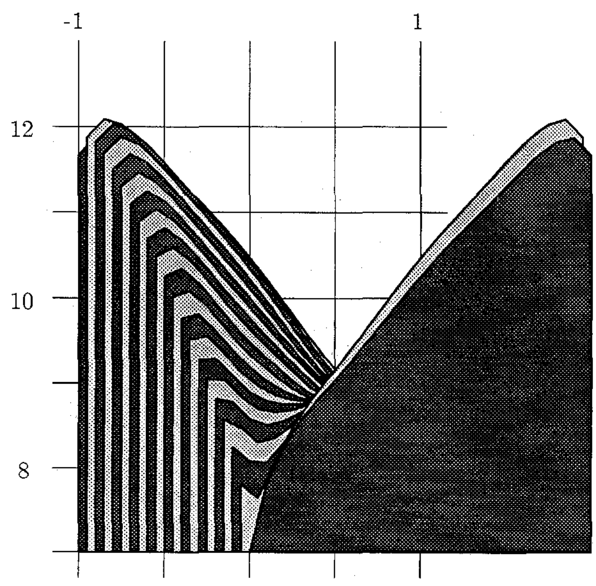

FIG. 3. Condition number of quartic B-splines with two interior knots.

```
% of the B-splines for the knot sequence t which has norm 1 on [t_k, t_n+1]
% and takes on the values 1 and -1 alternatingly the maximum possible number,
% i.e., n times, and is positive near t_n+1.

npk = length(t); n = npk - k;
t = [t(k) * ones(1, k), t(k + 1: n), t(n + 1) * ones(1, k)];
% here I omitted statements which would initialize rho as the average of
% k - 1 neighboring knots in case the initial guess provided is inadequate.

rho(1) = t(k); rho(n) = t(n + 1);     % the first and last rho are the endpoints of the
                                      % interval.
trho = rho(2: n - 1);                 % only the interior rho will be iterated on.
y = ones(rho);                        % set up the oscillating data to be matched by...
y(n - 1: -2: 1) = -y(n - 1: -2: 1);%  ... the Chebyshev spline.
change = 1; tsize = rho(n) - rho(1);  % set up convergence control.

iter = 0;
while (change > 1 . e - 8) & (iter < 8);
  sp = spcpi(t, rho, y); % compute the spline with knot sequence t which takes
                         % on the value y(j) at rho(j), j = 1, ..., n.
  dsp = spder(sp);       % construct the first derivative of this spline...
  drho = spval(dsp, trho); % ... and evaluate it at the interior rho.
  ddrho = spval(spder(dsp), trho); % also evaluate the second derivative of that
                                   % spline at the interior rho.
  drho = -drho ./ddrho; % compute the Newton step...
  trho = trho + drho;    % ... and add it to the current interior rho.
  % prevent modified rho from violating the expected interlacing by pulling
  % back on the proposed Newton step if necessary:
  count = 0;
  while (any(trho < t(3: n)) | any(trho > t(k + 1: n + k - 2)) | any(diff(trho) < =0)),
    drho = drho/2; trho = trho - drho;
    count = count + 1; if (count > 20), error ("no convergence"), end
  end
  change = max(abs(drho))/tsize % compute relative size of the step taken.
  rho(2: n - 1) = trho;         % update rho.
  iter = iter + 1;
end

[dummy, a] = spbrk(sp); % recover the B-spline coefficients a of the Chebyshev
                        % spline.
```

The calculations become quite delicate with increasing k and increasing nonuniformity of the knot sequence. I have not found a certain rule for choosing a satisfactory first guess, but have very often succeeded with the aid of continuation. For example, if the Chebyshev spline for the same (interior) knots but of one order lower is already available, then the midpoints between its neighboring extreme points often provide good first guesses for the interior extreme points of the Chebyshev spline to be computed.

Note that the "Chebyshev spline" used here is in general different from the "Chebyshev–Euler spline" used in Schoenberg and Cavaretta's solution

[SC] of the Landau problem on the halfline, and which also appears prominently in Tikhomirov's work (cf. [T]). The latter might be called "perfect Chebyshev splines" since they are Chebyshev splines whose highest nontrivial derivative is absolutely constant, a feat achieved only by an appropriate choice of knots. The more general Chebyshev splines of interest here have most recently appeared in Demko's [D] nice proof of the existence of "good" interpolation points for arbitrary knot sequences and, almost simultaneously, in Mørken [M], a reference of which I became aware only recently. Mørken devotes an entire chapter to the Chebyshev spline (which he calls, perhaps more helpfully, the "equioscillating spline"), proving its uniqueness by a detailed study of the sign structure. I note that uniqueness can also be deduced from the fact (mentioned earlier) that C_j is an extremal for λ_j.

THE 1-NORM CONDITION NUMBER

When the norm on $S = S_{k,\mathbf{t}}$ is the 1-norm,

$$\|f\|_1 := \int_{t_k}^{t_{n+1}} |f(t)|\, dt,$$

it is preferable to use also the 1-norm instead of the max-norm for the B-spline coefficients and to use a different normalization for the B-splines, too. Precisely, define

$$\mathrm{cond}^1_{k,\mathbf{t}} := \|\Phi\|\ \|\Phi^{-1}\|,$$

with

$$\Phi\colon l_1 \to S_{k,\mathbf{t}} \subset \mathbf{L}_1[t_k, t_{n+1}]\colon a \mapsto \sum_j M_j a(j)$$

and

$$M_j := M_{j,k,\mathbf{t}} := \frac{k}{t_{j+k} - t_j} N_{j,k,\mathbf{t}}.$$

Since $\|M_j\|_1 \leqslant 1$ for all j, we have

$$\|\Phi\| = \sup_a \frac{\|\sum M_j a(j)\|_1}{\|a\|_1} = 1,$$

hence

$$D_{k,1} := \sup_{\mathbf{t}} \mathrm{cond}^1_{k,\mathbf{t}} = \sup_{\mathbf{t}} \|\Phi^{-1}\|.$$

It is worthwhile to point out that, by duality, this number coincides with the *Favard constant* [B1]

$$K(k) := \sup_{f_0, \mathbf{t}} \frac{\inf\{\|D^k f\|_\infty : f \in \mathbf{L}_\infty^{(k)}, f = f_0 \text{ on } \mathbf{t}\}}{\max_i k!\, |[t_i, ..., t_{i+k}] f_0|}$$

which measures how small one can make the kth derivative of an interpolating function (relative to the kth divided differences $[t_i, ..., t_{i+k}] f_0$ of the given data). This fact has also been found by Otto [O], by rather different means.

LEMMA 2. $D_{k,1} = K(k)$.

Proof. Since $k![t_i, ..., t_{i+k}] f = \int M_i D^k f$, we can write $K(k)$ also as

$$K(k) = \sup_{\mathbf{t}} \sup_{g_0 \in \mathbf{L}_\infty} \frac{\inf\{\|g\|_\infty : \int M_j g = \int M_j g_0, \text{ all } j\}}{\max_j |\int M_j g_0|}.$$

This shows that

$$K(k) = \sup_{\mathbf{t}} \|F\|,$$

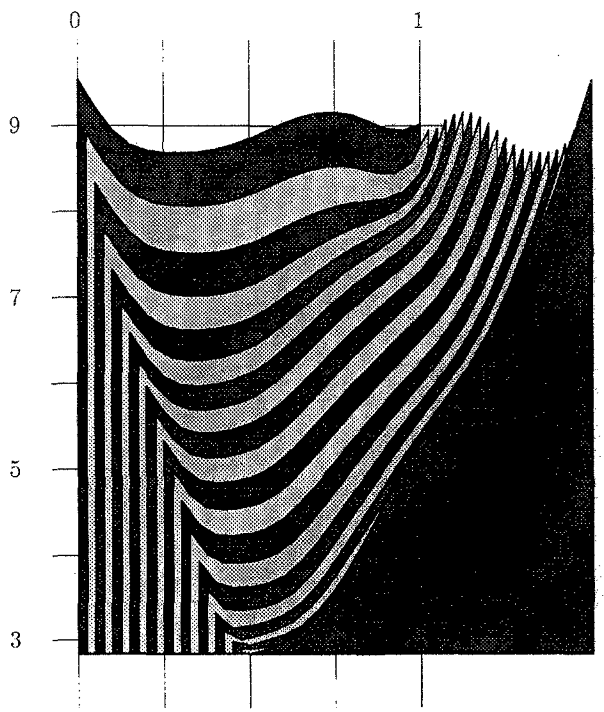

FIG. 4. $\|\lambda\|$ for $k = 5$ as a function of two interior knots.

with

$$F: l_\infty \to S^*: a \mapsto \sum \mu_i a(i)$$

and $\mu_i := \mu_{i,k,\mathbf{t}} := ((t_{i+k} - t_i)/k)\, \lambda_{i,k,\mathbf{t}}$ the linear functionals dual to the M_j's. But this implies that $F = (\Phi^*)^{-1}$ and, in particular, $\|F\| = \|\Phi^{-1}\|$. ▮

It follows that the calculations of $K(k)$ in [B1] are pertinent for the calculation of $D_{k,1}$. These calculations are based on the fact that

$$K(k) \leqslant 1 + 2(k-1) \sup_{\mathbf{s}} \|\lambda\|,$$

with $0 = s_1 = \cdots = s_k < s_{k+1} < \cdots < 1 = s_{2k-1} = \cdots = s_{3k-2}$ and λ the linear functional on $S_{k,\mathbf{s}} \subset \mathbf{L}_1[0,1]$ which carries $\sum_j M_j a(j)$ to $\sum_{j \geqslant k} a(j)$; see [B1] for details.

It is possible to compute $\|\lambda\|$ as a function of $(s_{k+j})_1^{k-2}$ by constructing the unique absolutely constant step function h on $[0, 1]$ with $2k-2$ jumps for which $\lambda f = \int hf$ for all $f \in S$. The calculations are almost identical to those reported in the final section. They show that, for small k, the supremum is achieved at one of the vertices of the domain over which the supremum is taken, i.e., when there are no interior knots. This is illustrated in Fig. 4 for $k = 5$.

The p-Norm Condition Number

Finally, consider the condition number of the B-spline basis when the norm on S is the p-norm,

$$\|f\|_p := \left(\int_{t_k}^{t_{n+1}} |f(t)|^p \, dt \right)^{1/p}.$$

It is shown in [B0] (see also [B2]) that

$$D_k^{-1} \|E^{1/p} a\|_p \leqslant \left\| \sum_j N_j a(j) \right\|_p \leqslant \|E^{1/p} a\|_p,$$

with

$$D_k := \sup_{\mathbf{t}} \sup_i k \, \|\mu_i\|^{(i)}, \qquad (E^\alpha a)(j) := (|I_j|/k)^\alpha \, a(j), \qquad \text{all } j,$$

and

$$\|\mu\|^{(i)} := \sup_f \frac{|\mu f|}{\int_{I_i} |f|}.$$

In particular, D_k is an upper bound for both $D_{k,\infty}$ and $D_{k,1}$, while also $kD_{k,1} \geqslant D_k$. Since $\mu_{j,\mathbf{t}} = ((t_{j+k} - t_j)/k)\, \lambda_{j,\mathbf{t}}$, it follows that

$$D_k = \sup_{\mathbf{s}} \|\lambda_{k,\mathbf{s}}\|,$$

with $s_1 = \cdots = s_k = 0 < s_{k+1} < \cdots < s_{2k-1} < 1 = s_{2k} = \cdots = s_{3k-1}$.

Explicitly,

$$D_k = \sup_{\mathbf{t}} \sup_i |I_i| \sup_f \frac{|\lambda_{i,\mathbf{t}} f|}{\int_{I_i} |f|}.$$

Hence, after a linear change of variables which carries the typical knot interval $I_i = [t_i, t_{i+k}]$ to the unit interval,

$$D_k = \sup_{\mathbf{s}} \sup_f \frac{|\lambda_{k,\mathbf{s}} f|}{\int_0^1 |f|},$$

with $0 = s_k \leqslant s_{k+1} \leqslant \cdots \leqslant s_{2k} = 1$. Since this involves the norm of

$$\lambda := \lambda_{k,\mathbf{s}}$$

on $S := S_{k,\mathbf{s}} \cap \mathbf{L}_1[0, 1]$, the knots s_j for $j < k$ or $j > 2k$ are immaterial; I take them to be 0 or 1, respectively. The remaining knots lie in $[0, 1]$.

Let $n := \dim S$. According to [B2], $\|\lambda\|$ is computable as the absolute height $\|h\|_\infty$ of the unique absolutely constant step function h on $[0, 1]$ with n steps which represents λ in the sense that

$$\lambda f = \int hf \qquad \text{for all} \quad f \in S.$$

If, more precisely, $0 = s_{k+r} < s_{k+r+1}$, then $\|h\|_\infty$ equals the norm of $\lambda = \lambda_{l,\mathbf{t}}$ on $S = S_{k,\mathbf{t}} \cap \mathbf{L}_1[0, 1]$, with

$$t_1 = \cdots = t_k = 0 < t_{k+1} \leqslant \cdots \leqslant t_n < 1 = t_{n+1} = \cdots = t_{n+k}$$

and

$$l = k - r.$$

The following MATLAB script returns this step function h for given l and given intt $:= (t_{k+r+1}, ..., t_n)$.

```
function [beta, tau, iter] = stepmk(left, intt, k, tau)
%
%       [beta, tau, iter] = stepmk(left, intt, k, [, tau])
%
% returns the absolutely constant step function with steps
```

```
% beta(i), i = 1, ..., n and breaks 0 = tau(1) < ··· < tau(n + 1) = 1 which
% represents the (left)th coordinate functional of the B-spline basis for the
% knot sequence t := [zeros(1, k) intt ones(1, k)], hence provides the norm of
% that functional wrto the 1-norm on [0, 1].
%
% On input, tau is assumed to contain a reasonable first guess.
% If missing or inappropriate, the knot averages are used.

tol = 1·e-4;
t = [zeros(1, k) intt ones(1, k)];
npk = length(t); n = npk - k;

% here I omitted statements which would initialize tau as the average of
% k neighboring knots in case the initial guess provided is inadequate.

dt = k * diag(ones(1, n)·/(t(1 + k: n + k) - t(1: n))); % matrix needed in the computation
                                                        % of change in tau.
b = zeros(n, 1); b(left) = k; b(left - 1) = -k; % generate the right side...
eps = ones(n, 1);                               % ... and a properly alternating
eps(1: 2: n) = -eps(1: 2: n);                   % ... sequence.
ttau = tau(2: n); gap = 1;

iter = 0;
while (0 = = 0);
  % generate the coefficient matrix. (Here, and below, spcol(s, k, tau) is the
  % matrix whose ith row consists of the values at tau(i) of all the
  % B-splines of order k for the knot sequence s, and diff(B) is the
  % matrix with entries B(i + 1, j) - B(i, j).)
  A = (diff(spcol([t, 1], k + 1, tau)))';

  beta = A\b; % compute the solution of the equation A * beta = b
  betamin = min(abs(beta)); % compute the relative nonconstancy.
  gap = (max(abs(beta)) - betamin)/betamin; %... of abs(beta) and...
  beta_gap = [beta(1), gap * 1·e+4] %... print it out, along with beta(1)

if (iter > 0) & ((gap < tol) | (iter > 10)), return; end
  % generate the change in tau:
  c = [spcol(t, k, ttau) * dt, zeros(n - 1, 1)];
  y = [-diff(c'), A * eps]\b;
  dtau = -(y(1: n - 1)./diff(beta))';
  ttau = ttau + dtau;

  % prevent changed tau from violating the expected interlacing by pulling
  % back on the Newton step if necessary:
  count = 0;
  while (any(ttau < t(1: n - 1)) | any(ttau > t(k + 2: n + k)) | any(diff(ttau) < =0)),
    dtau = dtau/2; ttau = ttau - dtau;
    count = count + 1; if (count > 20), error("no convergence"), end
  end
  tau = [0 ttau 1];
  iter = iter + 1;
end
```

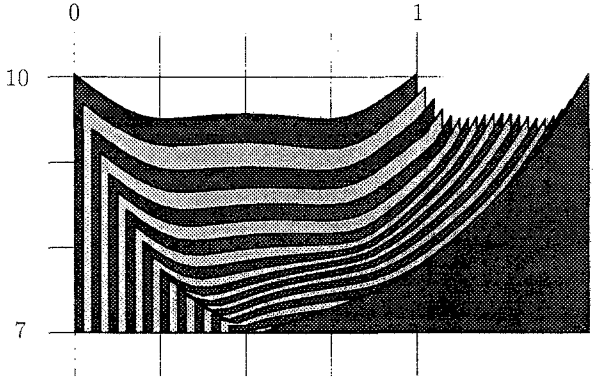

FIG. 5. $\|\lambda_{k,\mathbf{t}}\|$ for $k=4$ as a function of the two interior knots $(t_{k+1}, t_{k+2}) \in (0, 1)^2$.

Figures 5 and 6 parallel Fig. 2 and 3 and illustrate thereby that the function being maximized in order to obtain D_k does appear to be taking on that maximum at a "middle" vertex of the domain. Extensive calculations with the above script for $k \leqslant 21$ have not produced any counterexample to the conjecture that $\|\lambda_{k,\mathbf{s}}\|$ is maximized when $\mathbf{s}$ has no interior knots.

It is also evident that $\|\lambda_{k,\mathbf{s}}\|$ is minimized when $\mathbf{s}$ has just one interior knot, of maximal multiplicity, i.e., of multiplicity $k-1$. The characterization of $\|\lambda\|$ as the max-norm of λ's unique representer h in the form of an absolutely constant step function with n steps makes it easy to see that, in that case, the norm is independent of the location of that interior knot.

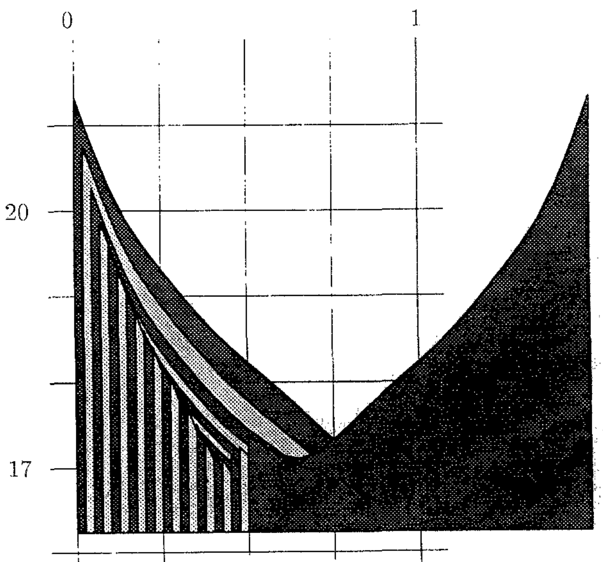

FIG. 6. $\|\lambda_{k-1,\mathbf{t}}\|$ for $k=5$ as a function of the two interior knots $(t_{k+1}, t_{k+2}) \in (0, 1)^2$.

Finally, the calculations of the representing step function h presented no numerical difficulties in all cases tried (up to $k = 21$), in marked contrast to the calculation of the Chebyshev splines.

CONCLUSION

There is numerical evidence that, in calculations devoted to bounding the p-norm condition number of the (appropriately scaled) B-spline basis, the extreme case occurs for a knot sequence without interior knots, while simple numerical examples show this not to be the case for the max-norm condition number itself. This is disappointing since it is only in the latter case that there seems to be a formula available for the condition number when there are no interior knots. Hence, even if the worst-case conjecture for the bound calculations for the p-norm condition number were proved, it would, offhand, not help in settling the problem of interest. This is the proof that all of these numbers, $D_{k,\infty}$, $D_{k,1} = K(k)$, and D_k, grow exactly like 2^k, as is suggested by numerical experiment.

Note added in proof. Following a suggestion from Phil Smith of IMSL, I have modified the iterative calculation of the Chebyshev spline (described in the MATLAB script chebmk above) to have the current extrema determined *locally*. This has made the calculations much less sensitive. See the Chebyshev spline example in the forthcoming "*Spline Toolbox*" to be published by MathWorks Inc., Sherborn, MA. This toolbox also contains the various spline routines referred to in the MATLAB script chebmk.

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